Point evaluation in Paley–Wiener spaces

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Abstract: For $0 < p < \infty$, the Paley–Wiener space PW^p is the subspace of $L^p(\mathbb{R})$ consisting of entire functions of exponential type at most π. We will discuss the following problem: What is the norm of point evaluation at the origin in PW^p? Thus we search for the smallest positive constant C, called $\mathscr{C}_p$, such that the inequality $|f(0)|^p \le C\|f\|_p^p$ holds for every f in PW^p, with $0 < p < \infty$.

This problem can be motivated in at least three different ways:

1. The quantity $\mathscr{C}_p^{-1}$ appears as a scaling limit for certain Christoffel functions;
2. How to compute $\mathscr{C}_p$ is a basic problem of time–frequency localization;
3. We are led to the following monotonicity conjecture: $p \mapsto \mathscr{C}_p/p$ *is strictly decreasing*. This can be thought of as a problem of how to do interpolation without interpolation theory being available.

In addition, the special case $p = 1$ has been studied by several authors owing largely to its relevance for problems in analytic number theory.

Discussing initially such motives, as well as some earlier contributions, we will present results from the recent paper [1]. Our approach is based on expressing $\mathscr{C}_p$ as the solution of an extremal problem, and we will see that the associated extremal functions, which exist for all $0 < p < \infty$, are real entire functions with only real and simple zeros. A certain Hilbert space structure associated with each extremal function gives us access to qualitative and quantitative information about the zeros, and we will explain how the latter kind of information enables us to obtain results that yield evidence in favor of the monotonicity conjecture mentioned above.

We will discuss in some detail a number of conjectures and further open problems pertaining to $\mathscr{C}_p$ and the associated extremal functions.

References

[1] O. F. Brevig, A. Chirre, J. Ortega-Cerdà, and K. Seip, *Point evaluation in Paley–Wiener spaces*, to appear in J. Anal. Math., arXiv:2210.13922v3.